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Multipliers from $L_1(G)$ to a Lipschitz Space

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Let G be a metric locally compact Abelian group. We prove that the spaces $(L_1, \mathrm{Lip}(\alpha, p))$, $(L_1, \mathrm{lip}(\alpha, p))$, $\mathrm{Lip}(\alpha, p)$ and $\mathrm{lip}(\alpha, p)^{\sim}$ are isometrically isomorphic, where $\mathrm{Lip}(\alpha, p)$ and $\mathrm{lip}(\alpha, p)$ denote the Lipschitz spaces defined on G, (L_1, A) is the space of multipliers from L_1 to A, and $\mathrm{lip}(\alpha, p)^{\sim}$ denotes the relative completion of $\mathrm{lip}(\alpha, p)$. We also show that $L_1 * \mathrm{Lip}(\alpha, p) = \mathrm{lip}(\alpha, p) = L_1 * \mathrm{lip}(\alpha, p)$.

1. INTRODUCTION AND PRELIMINARY RESULTS

Throughout this section G will denote a locally compact Abelian group with Haar measure λ. For $1 \leqslant p \leqslant \infty$, L_p will denote the usual Lebesgue space defined on G with respect to λ. In [2, Definition 3], Burnham defined the relative completion $\tilde{S}$ of a Segal algebra $S = S(G)$ to be the set of all $f \in L_1$ such that

$$f \in \bigcup_{\delta>0} \overline{B_S(\delta)},$$

where $B_S(\delta) = \{g \in S: \|g\|_S \leqslant \delta\}$ and $\bar{E}$ denotes the L_1-closure of E. For $f \in \tilde{S}$, define $\|f\|_{\tilde{S}} = \inf\{\delta: f \in \overline{B_S(\delta)}\}$. Burnham [2, Theorem 5] also proved that

$$\tilde{S} = \{f \in L_1: \sup_\beta \|f * e_\beta\|_S < \infty\}$$

and, for $f \in \tilde{S}$,

$$\|f\|_{\tilde{S}} = \sup_\beta \|f * e_\beta\|_S,$$

where $\{e_\beta\}$ is a common approximate identity for L_1 and S, and $\|e_\beta\|_1 = 1$ for all β. In [3, Theorem 2.6], Burnham and Goldberg proved that the space of multipliers from L_1 to a Segal algebra S is isometrically isomorphic to $\tilde{S}$, provided that every multiplier is absolutely continuous. In this section we prove a generalization of the Burnham–Goldberg result. This result is then applied, in Section 2, to obtain our result on multipliers of Lipschitz spaces (see Theo-

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rem 2.1). In Section 3 we derive our factorization theorem (see Theorem 3.1) from the theorem on multipliers.

The following definition is suggested by Burnham's characterization of the relative completion of a Segal algebra.

DEFINITION 1.1. Let A be a linear subspace of L_p, $1 \leqslant p < \infty$, with the following properties:

(M1) There is a norm $\| \ \|_A$ for A such that $\| \ \|_p \leqslant \| \ \|_A$ and $(A, \| \ \|_A)$ is a Banach L_1-module with respect to convolution.

(M2) There is an approximate identity $\{e_\beta\}$ in L_1 such that $\| e_\beta \|_1 = 1$ for all β, and $\| f * e_\beta - f \|_A \to 0$ for each $f \in A$.

The *relative completion* of A is the space

$$\tilde{A} = \{ f \in L_p : f * e_\beta \in A \text{ for all } \beta \text{ and } \sup_\beta \| f * e_\beta \|_A < \infty \}$$

with norm $\| \ \|_{\tilde{A}}$ defined by $\| f \|_{\tilde{A}} = \sup_\beta \| f * e_\beta \|_A$.

The easy proof of the following lemma will be omitted.

LEMMA 1.2. *Let A be as in Definition* 1.1. *Then we have*:

(i) *If $f \in \tilde{A}$ and $g \in L_1$, then $f * g \in \tilde{A}$ and*

$$\| f * g \|_{\tilde{A}} \leqslant \| f \|_{\tilde{A}} \| g \|_1 .$$

(ii) $\| f \|_A = \| f \|_{\tilde{A}}$ *for $f \in A$.*

(iii) *A is a closed subspace of $\tilde{A}$.*

Before we proceed further, we recall that by a multiplier from L_1 to A we mean a bounded linear mapping $T: L_1 \to A$ that commutes with translations. The set of multipliers from L_1 to A will be denoted by (L_1, A). The following theorem is suggested by Burnham–Goldberg [3, Theorem 2.6] where the case $p = 1$ is considered.

THEOREM 1.3. *Let A be a linear subspace of L_p, $1 \leqslant p < \infty$, satisfying Properties* (M1) *and* (M2). *Then (L_1, A) is isometrically isomorphic to $\tilde{A}$ if $p > 1$, or if $p = 1$ and $(L_1, A) \subset L_1$.*

Proof. Let $f \in \tilde{A}$. Define a mapping T_f on L_1 by $T_f(g) = f * g$. We will show that $T_f \in (L_1, A)$ and $\| T_f \| = \| f \|_{\tilde{A}}$.

Let $g \in L_1$. It follows from Lemma 1.2 that

$$\| f * g * e_{\beta_1} - f * g * e_{\beta_2} \|_A \leqslant \| f \|_{\tilde{A}} \| g * e_{\beta_1} - g * e_{\beta_2} \|_1 .$$

Thus $\{f * g * e_\beta\}$ is a Cauchy net in A and so it converges to some h in A. By Lemma 1.2, $f * g \in \tilde{A}$ and

$$\begin{aligned}\|f * g - h\|_{\tilde{A}} &\leqslant \|f * g - f * g * e_\beta\|_{\tilde{A}} + \|f * g * e_\beta - h\|_{\tilde{A}} \\ &\leqslant \|f\|_{\tilde{A}} \|g - g * e_\beta\|_1 + \|f * g * e_\beta - h\|_A \to 0.\end{aligned}$$

This shows that $T_f(g) = f * g = h \in A$. Thus T_f is a mapping from L_1 to A. Clearly T is linear and it commutes with translations. It follows from Lemma 1.2 that $T_f \in (L_1, A)$ and $\| T_f \| \leqslant \|f\|_{\tilde{A}}$. Since $\| T_f \| \geqslant \|f\|_{\tilde{A}}$ is clearly true, we have $\| T_f \| = \|f\|_{\tilde{A}}$.

Conversely, let $T \in (L_1, A)$. Then (by Larsen [7, Theorem 3.1.1] for the case $p > 1$ and by our hypothesis for the case $p = 1$) there exists $f \in L_p$ such that $Tg = f * g$ for all $g \in L_1$. It follows that

$$\sup_\beta \|f * e_\beta\|_A \leqslant \| T \|,$$

and so $f \in \tilde{A}$. Thus every $T \in (L_1, A)$ is of the form T_f for some $f \in \tilde{A}$ with $\| T_f \| = \|f\|_{\tilde{A}}$. Hence the mapping $f \to T_f$ is an isometric isomorphism from $\tilde{A}$ onto (L_1, A).

Remark. Another extension of the Burnham–Goldberg result is given in Goldberg–Seltzer [5]. However, in a forthcoming paper [4], Feichtinger has obtained results which subsume both the Goldberg–Seltzer result and Theorem 1.3. We have included Theorem 1.3 and its simple proof for the convenience of the reader.

2. Multipliers of Lipschitz Spaces

For the remainder of this note we will assume that the underlying group G is a metric locally compact Abelian group with translation-invariant metric d. We will also assume that there is a decreasing countable (open) basis $\{V_n\}$ at the identity 0 in G such that

$$\lambda(y + V_n \triangle V_n)/d(0, y)^\alpha \to 0 \qquad \text{as} \quad y \to 0,$$

where $\triangle$ denotes the symmetric difference, and $0 < \alpha < 1$. The following remark shows that the above condition is not unduly restrictive.

Remark. (i) In [8, Lemma 1], Struble proved that if $\{V_n\}$ is any decreasing countable (open) basis at 0 with V_1 precompact, then

$$\rho(x, y) = \sup_n \lambda(x + V_n \triangle y + V_n)$$

defines a translation-invariant metric in G which is equivalent to the given metric d in G. It is clear that $\lambda(y + V_n \triangle V_n)/\rho(0, y)^\alpha \to 0$ as $y \to 0$.

(ii) If G is 0-dimensional, then there is a basis $\{V_n\}$ at 0 consisting of a decreasing sequence of compact open subgroups of G (see [6, 7.7]). Hence we have $\lambda(y + V_n \triangle V_n)/d(0, y)^\alpha \to 0$ as $y \to 0$ for any metric d in G.

(iii) For $G = R^k$ or T^k $(k \geqslant 1)$, define $V_n = (-1/n, 1/n) \times \cdots \times (-1/n, 1/n)$ (k times) for $n = 1, 2, 3,\dots$. Now let $y = (y_1, \dots, y_k)$ and $y^{(j)} = (0,\dots, 0, y_j, 0,\dots, 0)$, and define $V_n^{(0)} = V_n$ and $V_n^{(j)} = y^{(j)} + V_n^{(j-1)}$ for $j = 1, 2,\dots, k$. Note that $V_n^{(k)} = y + V_n$. It is easy to verify that

$$y + V_n \triangle V_n \subseteq \bigcup_{j=1}^{k} (V^{(j)} \triangle V^{(j-1)}),$$

and hence

$$\lambda(y + V_n \triangle V_n) \leqslant \sum_{j=1}^{k} 2(2/n)^{k-1} \mid y_j \mid \leqslant 2^k n^{1-k} k^{1/2}\, d(0, y),$$

where d denotes the Euclidean metric. Thus we have

$$\lambda(y + V_n \triangle V_n)/d(0, y)^\alpha \to 0 \qquad \text{as} \quad y \to 0.$$

We will assume from now on that $1 \leqslant p < \infty$ and $0 < \alpha < 1$. For $y \in G$, let $\mid y \mid = d(0, y)$. $\{V_n\}$ will denote a decreasing countable (open) basis at 0 such that $\lambda(y + V_n \triangle V_n)/\mid y \mid^\alpha \to 0$ as $y \to 0$, and $\{e_n\}$ will denote the approximate identity in $L_1(G)$ defined by $e_n = \lambda(V_n)^{-1} \chi_{V_n}$, where χ denotes the characteristic function.

For $f \in L_p$ and $\delta > 0$, define

$$\omega_p(f; \delta) = \sup\{\| T_y f - f \|_p : \mid y \mid \leqslant \delta\},$$

where $T_y f(x) = f(x - y)$. Following Zygmund [9, p. 45] and Bloom [1, p. 150], we define

$$\begin{aligned} \operatorname{Lip}(\alpha, p) &= \{f \in L_p : \omega_p(f; \delta) = O(\delta^\alpha)\}, \\ \operatorname{lip}(\alpha, p) &= \{f \in \operatorname{Lip}(\alpha, p) : \omega_p(f; \delta) = o(\delta^\alpha)\}. \end{aligned}$$

It is clear that the function $\| \; \|_{(\alpha, p)}$ defined by

$$\| f \|_{(\alpha, p)} = \| f \|_p + \sup_{y \neq 0} \frac{\| T_y f - f \|_p}{\mid y \mid^\alpha}$$

is a norm in $\operatorname{Lip}(\alpha, p)$.

The main result of this paper can now be stated as follows.

THEOREM 2.1. *The spaces* $(L_1, \operatorname{lip}(\alpha, p))$, $\operatorname{lip}(\alpha, p)^\sim$, $\operatorname{Lip}(\alpha, p)$ *and* $(L_1,$

$\mathrm{Lip}(\alpha, p)$) *are isometrically isomorphic, where* $\mathrm{lip}(\alpha, p)^{\sim}$ *denotes the relative completion of* $\mathrm{lip}(\alpha, p)$.

This theorem will be deduced from a series of lemmas.

LEMMA 2.2. *If* $f \in \mathrm{lip}(\alpha, p)$, *then* $\| T_x f - f \|_{(\alpha,p)} \to 0$ *as* $x \to 0$.

Proof. Let $\epsilon > 0$. Since $f \in \mathrm{lip}(\alpha, p)$, there exists $\delta > 0$ such that

$$\sup_{|y| \leqslant c} \frac{\| T_y f - f \|_p}{c^\alpha} < \frac{\epsilon}{2}$$

for $0 < c < \delta$. Thus

$$\frac{\| T_y f - f \|_p}{| y |^\alpha} < \frac{\epsilon}{2} \qquad \text{for } 0 < | y | < \delta. \tag{1}$$

It is easy to see that for any $x, y \in G$, we have

$$\| T_y(T_x f - f) - (T_x f - f) \|_p \leqslant 2 \| T_y f - f \|_p . \tag{2}$$

It follows that if x is in G, then

$$\begin{aligned} \| T_x f - f \|_{(\alpha,p)} &= \| T_x f - f \|_p + \sup_{y \neq 0} \frac{\| T_y(T_x f - f) - (T_x f - f) \|_p}{| y |^\alpha} \\ &= \| T_x f - f \|_p + \sup_{|y| < \delta} \frac{\| T_y(T_x f - f) - (T_x f - f) \|_p}{| y |^\alpha} \\ &\qquad + \sup_{|y| \geqslant \delta} \frac{\| T_y(T_x f - f) - (T_x f - f) \|_p}{| y |^\alpha} \\ &\leqslant \| T_x f - f \|_p + \epsilon + \frac{2 \| T_x f - f \|_p}{\delta^\alpha} \qquad \text{(by (1) and (2))}. \end{aligned}$$

Lemma 2.2 now follows from the fact that $\| T_x f - f \|_p \to 0$ as $x \to 0$.

LEMMA 2.3. *If* $f \in \mathrm{lip}(\alpha, p)$, *then* $\| f * e_n - f \|_{(\alpha,p)} \to 0$.

Proof. For all n, we have

$$f * e_n - f = \int e_n(y) \, (T_y f - f) \, dy$$

and so

$$\| f * e_n - f \|_{(\alpha,p)} \leqslant \int e_n(y) \, \| T_y f - f \|_{(\alpha,p)} \, dy.$$

The conclusion follows from this inequality and Lemma 2.2.

LEMMA 2.4. $(\mathrm{lip}(\alpha, p), \| \ \|_{(\alpha,p)})$ *is a Banach* L_1*-module with respect to convolution.*

Proof. It suffices to show that $\mathrm{lip}(\alpha, p)$ is complete with respect to the norm $\| \ \|_{(\alpha,p)}$, since the other properties are obvious.

Let $\{f_i\}$ be a Cauchy sequence in $\mathrm{lip}(\alpha, p)$. Then there exists $f \in L_p$ such that $\|f_i - f\|_p \to 0$. We will show that $f \in \mathrm{lip}(\alpha, p)$ and $f_i \to f$ in $\mathrm{lip}(\alpha, p)$.

Let $\epsilon > 0$. Then there exists a positive integer N such that

$$\begin{aligned} \|f_n - f\|_p &< \frac{\epsilon}{4} \qquad \text{for all } n \geqslant N, \\ \|f_n - f_j\|_{(\alpha,p)} &< \frac{\epsilon}{4} \qquad \text{for all } n, j \geqslant N. \end{aligned} \tag{1}$$

For $y \in G$ with $y \neq 0$, choose a positive integer $k \geqslant N$ (k depends on y) such that

$$\|f_k - f\|_p \leqslant \frac{\epsilon}{4} |y|^\alpha.$$

Thus, for $n \geqslant N$, we have

$$\begin{aligned} \|T_y(f_n - f) - (f_n - f)\|_p &\leqslant \|T_y(f_n - f_k) - (f_n - f_k)\|_p + \|T_y(f_k - f)\|_p \\ &\quad + \|f - f_k\|_p \\ &\leqslant \frac{3\epsilon}{4} |y|^\alpha. \end{aligned} \tag{2}$$

Hence

$$\begin{aligned} \frac{\|T_y f - f\|_p}{|y|^\alpha} &\leqslant \frac{\|T_y(f - f_N) - (f - f_N)\|_p}{|y|^\alpha} + \frac{\|T_y f_N - f_N\|_p}{|y|^\alpha} \\ &\leqslant \frac{3}{4}\epsilon + \frac{\|T_y f_N - f_N\|_p}{|y|^\alpha}. \end{aligned} \tag{3}$$

Since $f_N \in \mathrm{lip}(\alpha, p)$, it follows from (3) that $f \in \mathrm{lip}(\alpha, p)$. By (1) and (2) we have $\|f_n - f\|_{(\alpha,p)} < \epsilon$ for $n \geqslant N$, and so $f_n \to f$ in $\mathrm{lip}(\alpha, p)$.

LEMMA 2.5. *The relative completion* $\mathrm{lip}(\alpha, p)^\sim$ *of* $\mathrm{lip}(\alpha, p)$ *is* $\mathrm{Lip}(\alpha, p)$.

Proof. Let $f \in \mathrm{lip}(\alpha, p)^\sim$ and let $\|f\|_{(\alpha,p)\sim}$ denote its norm (i.e., $\|f\|_{(\alpha,p)\sim} = \sup \|f * e_n\|_{(\alpha,p)}$). Clearly

$$\|T_y(f * e_n) - (f * e_n)\|_p \leqslant \|f\|_{(\alpha,p)\sim} |y|^\alpha$$

for all n and for all $y \neq 0$. Now note that we have

$$\| T_y f - f \|_p \leqslant \| T_y f - T_y(f * e_n)\|_p + \| T_y(f * e_n) - f * e_n \|_p + \| f * e_n - f \|_p$$

for all n. Since $f * e_n \to f$ in L_p, we have

$$\| T_y f - f \|_p \leqslant \| f \|_{(\alpha, p)^\sim} \, | y |^\alpha$$

and so $f \in \mathrm{Lip}(\alpha, p)$. This shows that $\mathrm{lip}(\alpha, p)^\sim \subset \mathrm{Lip}(\alpha, p)$.

Since $\| T_y e_n - e_n \|_1 = \lambda(V_n)^{-1} \lambda(y + V_n \triangle V_n)$, it is clear that $e_n \in \mathrm{lip}(\alpha, 1)$. Now if $f \in \mathrm{Lip}(\alpha, p)$, then it follows from $\| T_y(f * e_n) - f * e_n \|_p \leqslant \| f \|_p \| T_y e_n - e_n \|_1$ that $f * e_n \in \mathrm{lip}(\alpha, p)$. Thus the sets $\mathrm{Lip}(\alpha, p)$ and $\mathrm{lip}(\alpha, p)^\sim$ are equal.

Next we show that $\| \; \|_{(\alpha, p)} \leqslant \| \; \|_{(\alpha, p)^\sim}$. Let $f \in \mathrm{Lip}(\alpha, p)$. Let $\epsilon > 0$. For $y \in G$ with $y \neq 0$, there exists n_0 such that

$$\| f - f * e_{n_0} \|_p < \frac{\epsilon}{2} \qquad \text{and} \qquad \|(T_y f - f) - (T_y f - f) * e_{n_0} \|_p < \frac{\epsilon \, | y |^\alpha}{2}.$$

Thus we have

$$\| f \|_p < \frac{\epsilon}{2} + \| f * e_{n_0} \|_p$$

and

$$\| T_y f - f \|_p < \frac{\epsilon \, | y |^\alpha}{2} + \|(T_y f - f) * e_{n_0} \|_p .$$

Hence

$$\begin{aligned} \| f \|_p + \frac{\| T_y f - f \|_p}{| y |^\alpha} &\leqslant \epsilon + \| f * e_{n_0} \|_p + \frac{\|(T_y f - f) * e_{n_0} \|_p}{| y |^\alpha} \\ &\leqslant \epsilon + \| f * e_{n_0} \|_{(\alpha, p)} \\ &\leqslant \epsilon + \| f \|_{(\alpha, p)^\sim} . \end{aligned}$$

Thus $\| \; \|_{(\alpha, p)} \leqslant \| \; \|_{(\alpha, p)^\sim}$. Since the reverse inequality is obvious, the two norms are equal.

LEMMA 2.6. $(L_1, \mathrm{Lip}(\alpha, p)) \subset \mathrm{Lip}(\alpha, p)$.

Proof. Let $T \in (L_1, \mathrm{Lip}(\alpha, p))$. Then $T \in (L_1, L_p)$. Thus if $p > 1$, then there exists $f \in L_p$ such that $Tg = f * g$ for all $g \in L_1$. Now suppose $p = 1$. Then there exists $\mu \in M(G)$ such that $Tg = \mu * g$ for all $g \in L_1$. We assert that $\mu \in L_1$.

Since T is bounded, there is a constant C such that $\| Tg \|_{(\alpha,1)} = \| \mu * g \|_{(\alpha,1)} \leqslant C \| g \|_1$ for all $g \in L_1$. It follows that $\| \mu * e_n \|_{(\alpha,1)} \leqslant C$, and so

$$\| T_y(\mu * e_n) - (\mu * e_n) \|_1 \leqslant C \mid y \mid^\alpha$$

for all $y \in G$ and for all n. Now let $\epsilon > 0$ and define $V = \{y \in G: \mid y \mid^\alpha \leqslant \epsilon/C\}$. Then there exists n_0 such that $V_{n_0} \subset V$. Hence, for all n, we have

$$\| \mu * e_n * e_{n_0} - \mu * e_n \|_1 \leqslant \int e_{n_0}(y) \| T_y(\mu * e_n) - \mu * e_n \|_1 \leqslant \epsilon. \qquad (1)$$

Since $\mu * e_{n_0} \in L_1$ and $\{e_n\}$ is an approximate identity in L_1, $\{\mu * e_{n_0} * e_n\}$ is a Cauchy sequence. Hence, by (1) above, $\{\mu * e_n\}$ is also a Cauchy sequence in L_1. Let $\mu * e_n \to f$ in L_1. Thus $\hat{\mu}\hat{e}_n \to \hat{f}$ uniformly on the dual group of G. But $\hat{\mu}\hat{e}_n \to \hat{\mu}$. Thus $\hat{f} = \hat{\mu}$, and so $\mu = f \in L_1$.

Thus for $p \geqslant 1$, there exists $f \in L_p$ such that $Tg = f * g$ for all $g \in L_1$. It is now clear that $\sup \| f * e_n \|_{(\alpha,p)} \leqslant C$, and so $f \in \text{lip}(\alpha, p)^\sim$. By Lemma 2.5, $f \in \text{Lip}(\alpha, p)$.

Proof of Theorem 2.1. The proof of Lemma 2.6 tells us that $(L_1, \text{lip}(\alpha, 1)) \subset L_1$. It follows from Lemma 2.4 and Theorem 1.3 that $(L_1, \text{lip}(\alpha, p)) = \text{lip}(\alpha, p)^\sim$. Hence, by Lemmas 2.5 and 2.6, we have

$$\text{Lip}(\alpha, p) = \text{lip}(\alpha, p)^\sim = (L_1, \text{lip}(\alpha, p)) \subset (L_1, \text{Lip}(\alpha, p)) \subset \text{Lip}(\alpha, p)$$

and so the four sets are equal. By Theorem 1.3 and Lemma 2.5 we see that the four spaces are isometrically isomorphic.

3. A Factorization Theorem

By using results of the preceding section the following theorem can now be proved easily.

THEOREM 3.1. $L_1 * \text{lip}(\alpha, p) = \text{lip}(\alpha, p) = L_1 * \text{Lip}(\alpha, p)$.

Proof. Let $f \in \text{lip}(\alpha, p)$ and let $\epsilon > 0$. By Lemma 2.2 there exists n_0 such that $\| T_y f - f \|_{(\alpha,p)} \leqslant \epsilon$ for all $y \in V_{n_0}$. It follows that

$$\| f * e_{n_0} - f \|_{(\alpha,p)} \leqslant \int e_{n_0}(y) \| T_y f - f \|_{(\alpha,p)} \, dy \leqslant \epsilon.$$

Hence $L_1 * \text{lip}(\alpha, p)$ is dense in $\text{lip}(\alpha, p)$. By [6, 32.22] we have $L_1 * \text{lip}(\alpha, p) =$

$\mathrm{lip}(\alpha, p)$. By Theorem 2.1 we have $(L_1, \mathrm{lip}(\alpha, p)) = \mathrm{Lip}(\alpha, p)$. It follows that $L_1 * \mathrm{Lip}(\alpha, p) \subset \mathrm{lip}(\alpha, p)$ and hence

$$\mathrm{lip}(\alpha, p) = L_1 * \mathrm{lip}(\alpha, p) \subset L_1 * \mathrm{Lip}(\alpha, p) \subset \mathrm{lip}(\alpha, p).$$

Therefore $L_1 * \mathrm{Lip}(\alpha, p) = \mathrm{lip}(\alpha, p)$.

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